On the Computational Complexity of Linear Discrepancy

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University of Toronto

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A				
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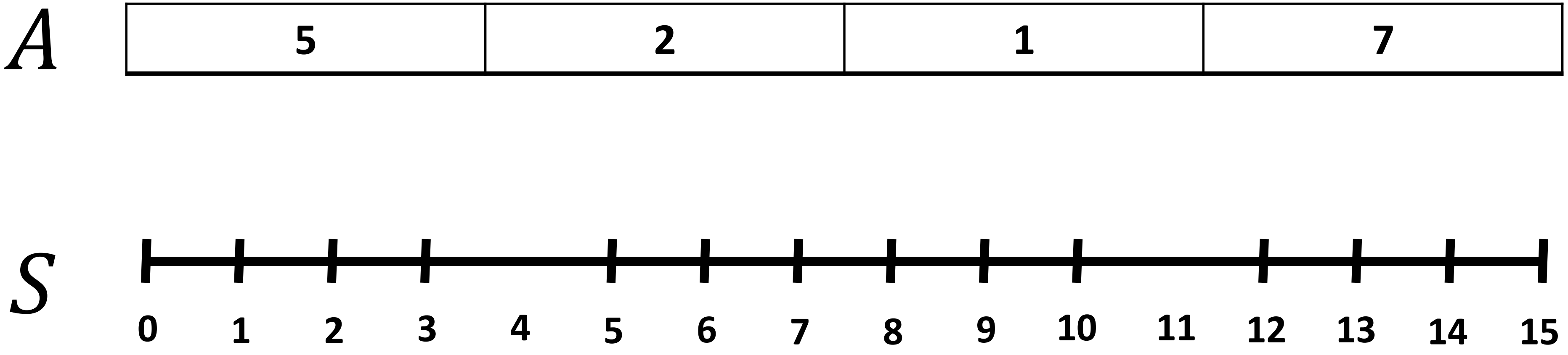
[OUT] Let S be the array of subset sums of A arranged in increasing order.



[IN] An array A of n real numbers.

[OUT] Let S be the array of subset sums of A arranged in increasing order. Find <u>*the largest gap*</u> $[s_i, s_{i+1}]$ between consecutives terms in S.

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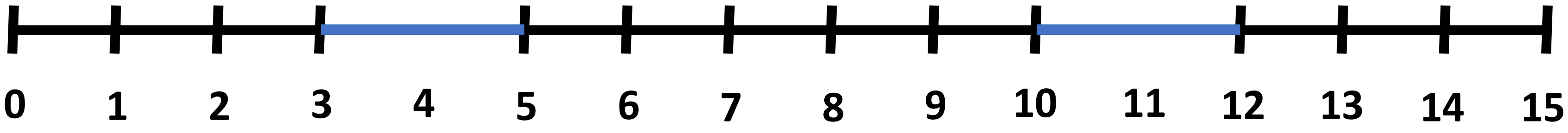
S



Point Distributions

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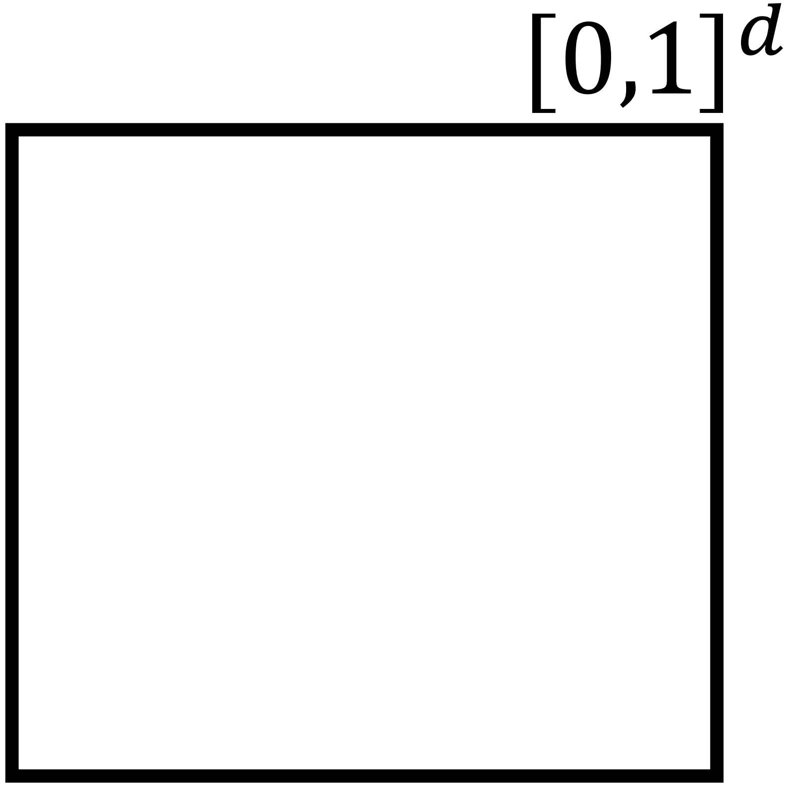
[IN] All axis-aligned boxes in $[0,1]^d$, denoted $\mathcal{R}_d$, and Lebesgue measure λ_d on $\mathbb{R}^d$.



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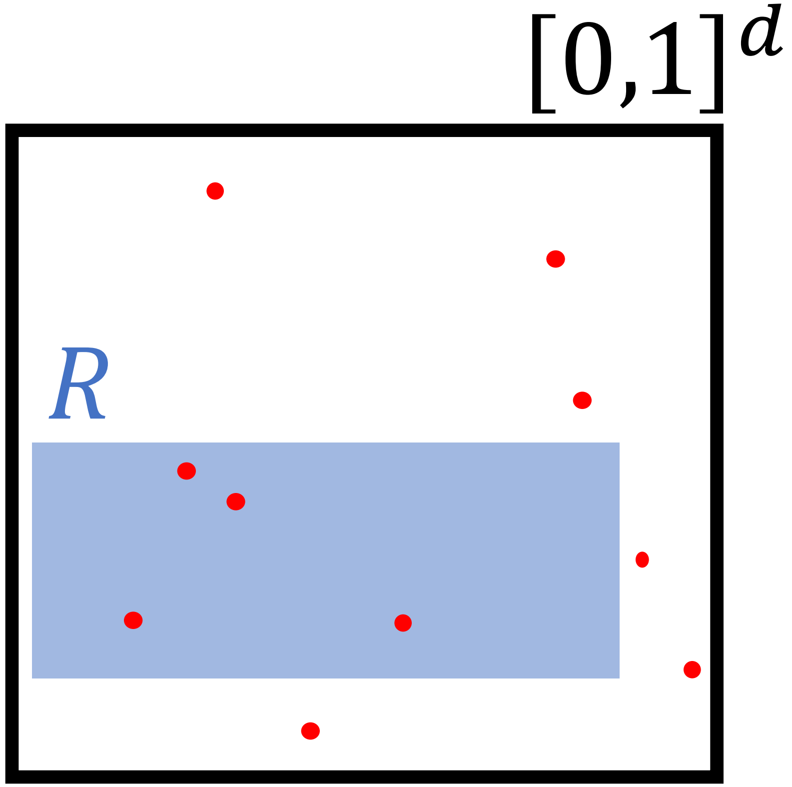
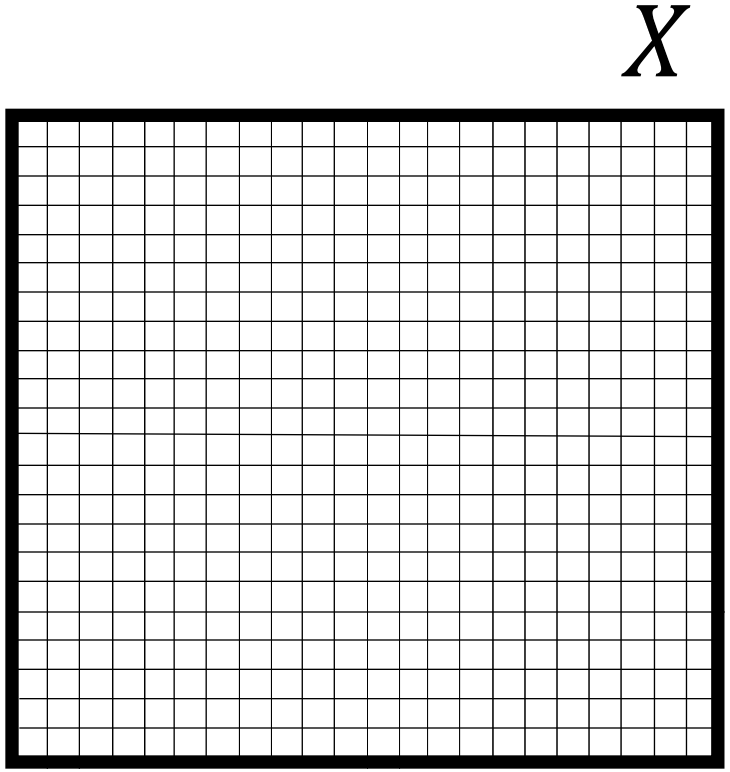
[OUT] Point set $P \in [0,1]^d$ which minimizes $\sup_{R\in\mathcal{R}_d} \left| |R \cap P| - n\lambda_d(R) \right|$.



Point Distributions

Discretize $[0,1]^d$ to finite set X of size N.



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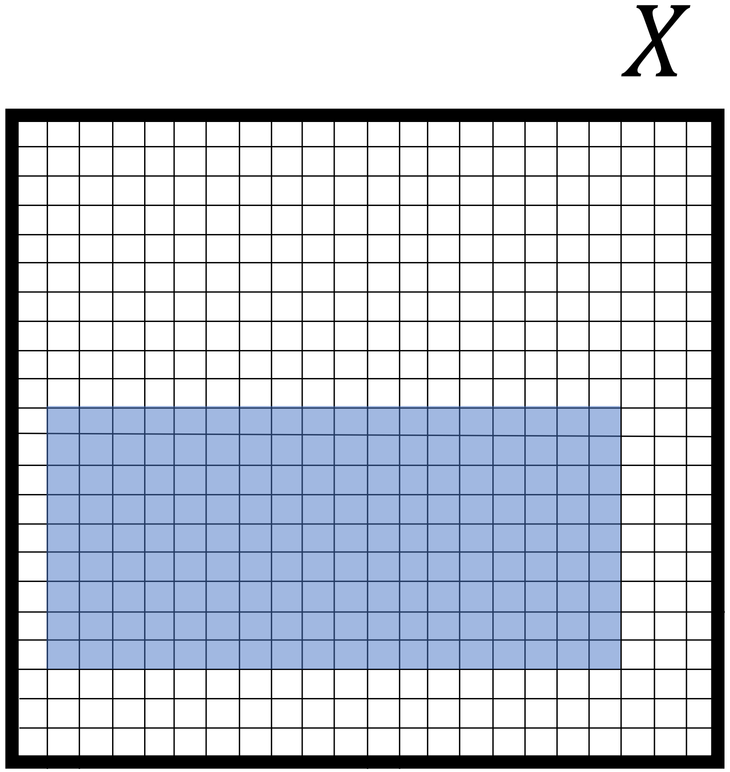

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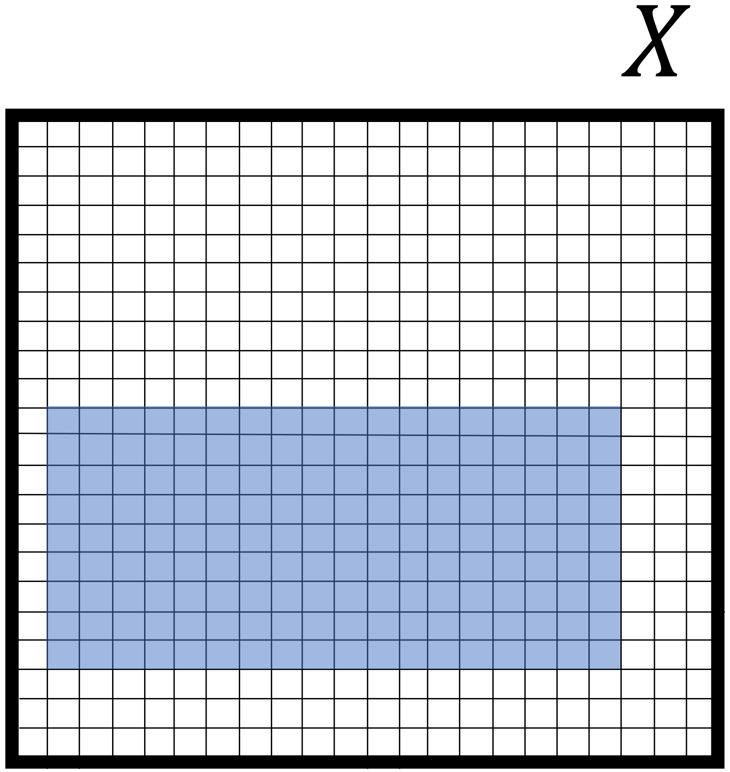

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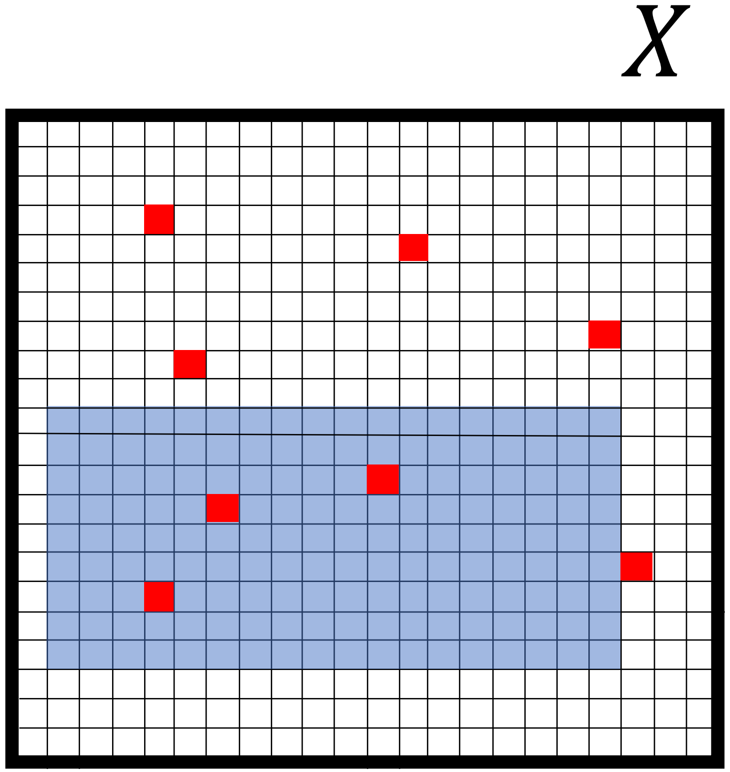

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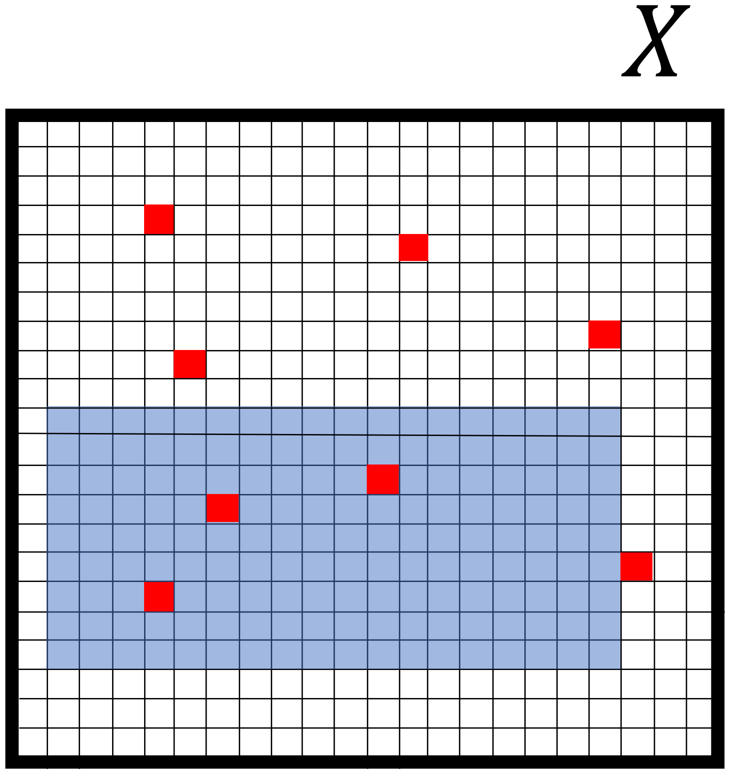

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$$\max c^{\top} w$$
$$Mw \leq b$$
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ILP (Relax- -)

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ILP (Relax-Solve-)

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$$\min_{x\in\{0,1\}^n} \max_{i\in[m]} \underbrace{|(Mx - Mw)_i|}_{\text{rounded int solution} \; - \; \text{LP solution}} = \min_{x\in\{0,1\}^n} \|M(x-w)\|_\infty$$

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Linear Discrepancy of matrix $M \in \mathbb{R}^{m\times n}$:

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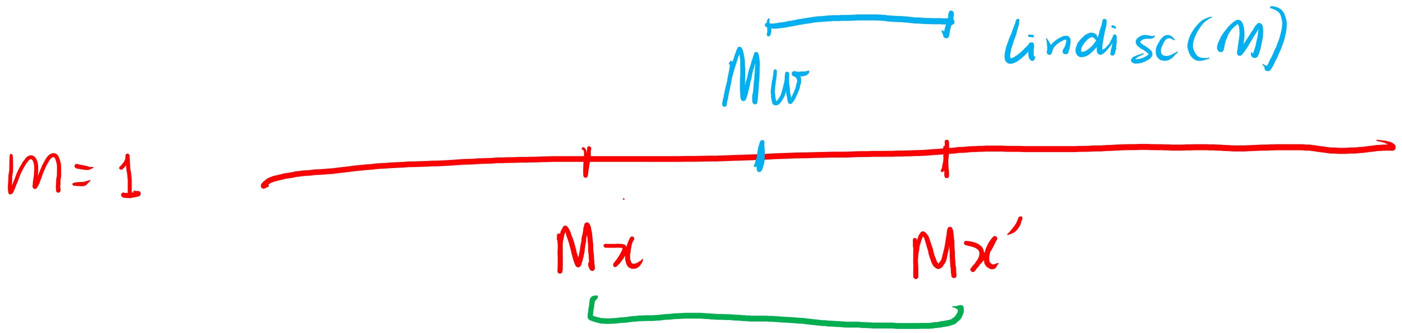

Linear Discrepancy of matrix $M \in \mathbb{R}^{m \times n}$:

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- $M \in \mathbb{Z}^{d\times n}$ with entries of magnitude bounded by δ; exact linear discrepancy in time
$$O\left(d(n\delta)^{d^2+d}\right)$$
- When $d = 1$, exact linear discrepancy can be computed in $O(n\log n)$.
- Linear discrepancy is **NP-Hard**
- $M \in \mathbb{Z}^{m\times n}$; 2^{n+1} approximation in polynomial time

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$$lindisc(w, M) = \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty \geq_p \text{Subset Sum}$$

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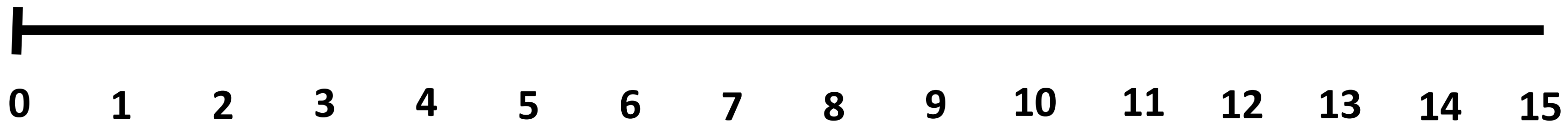
$k = 0 \qquad 2lindisc(A_0) = 0$

A

7	**5**	**2**	**1**

S



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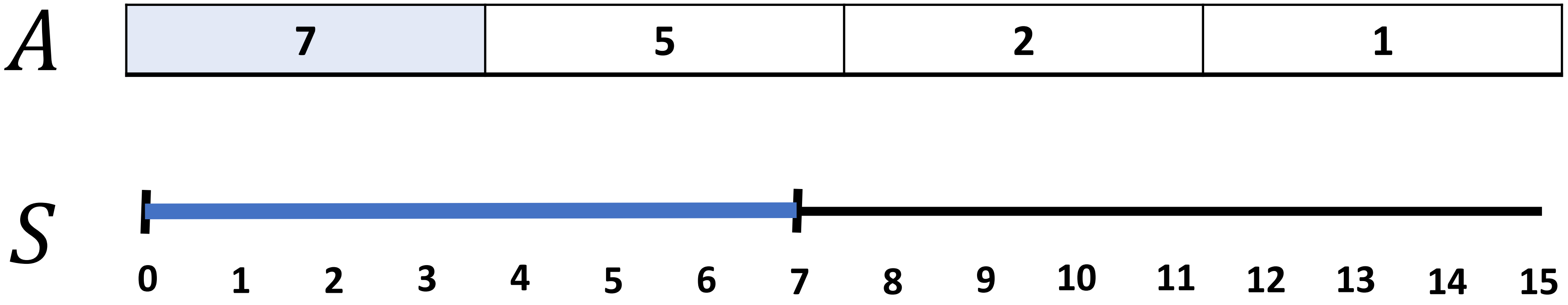
$k = 1 \qquad 2lindisc(A_1) = 7$



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$k = 1 \qquad 2lindisc(A_1) = 7$

$k = 2 \qquad 2lindisc(A_2) = 5$

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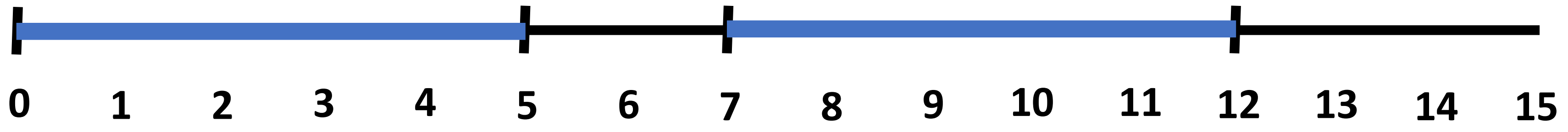

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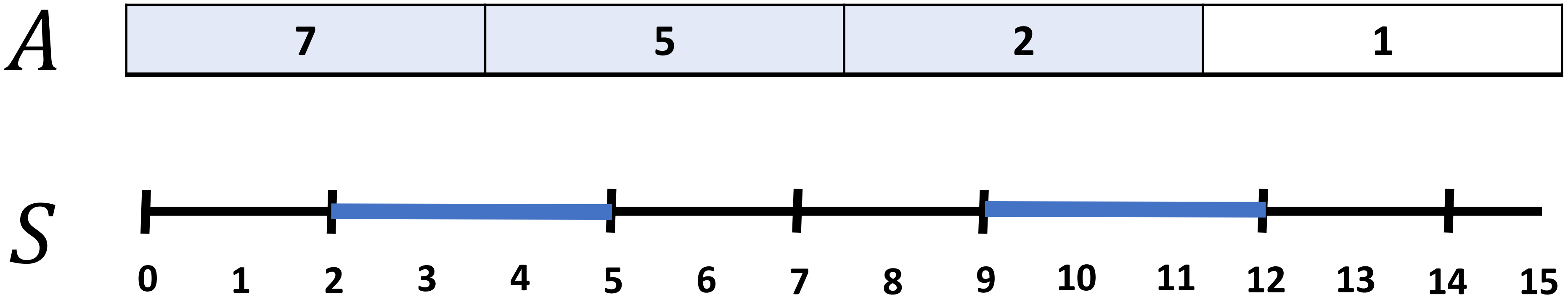

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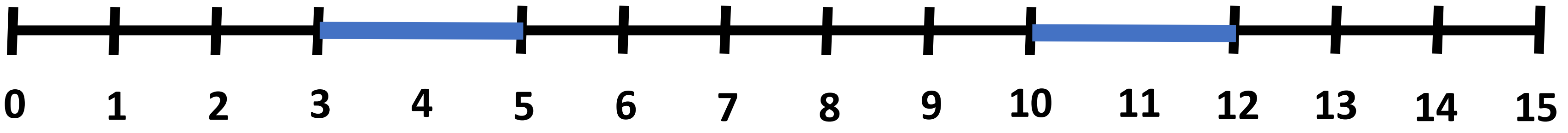

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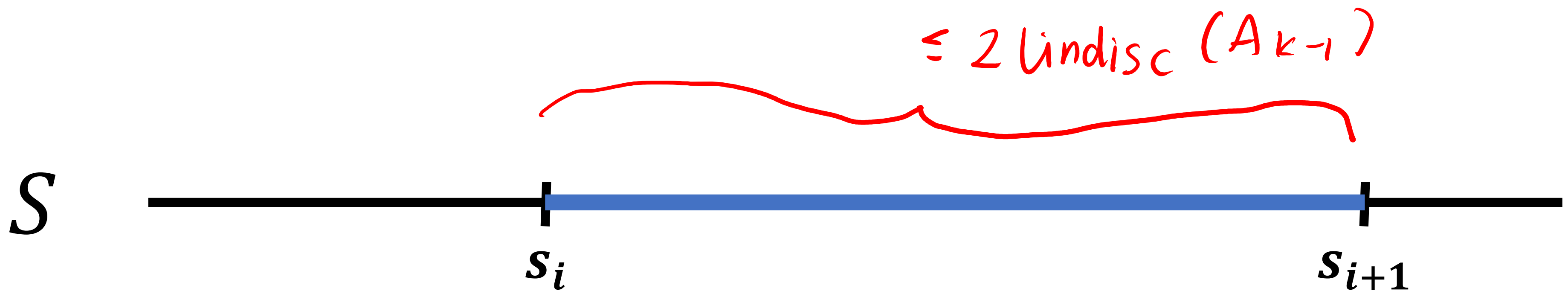

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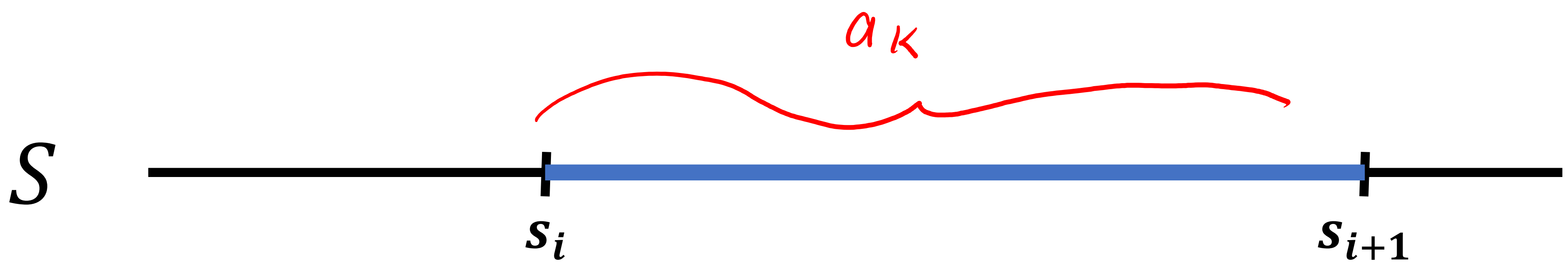

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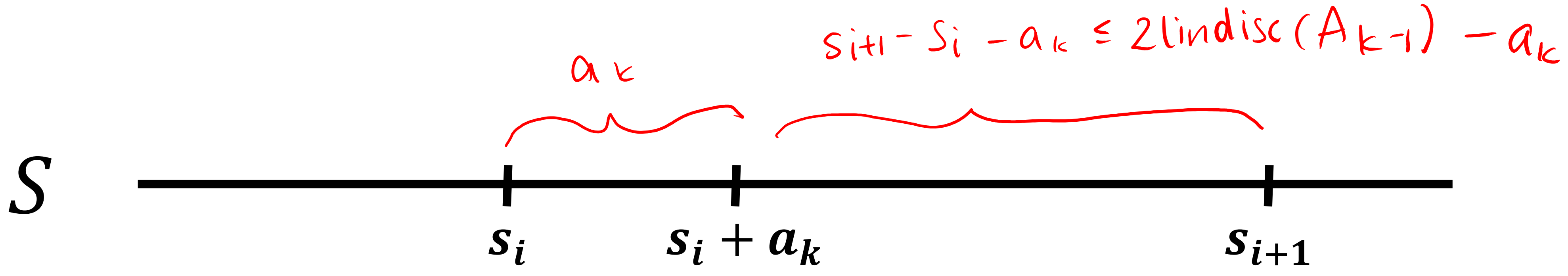

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Two cases: (i) $a_k \geq 2lindisc(A_{k-1}) - a_k$, or (ii) $a_k < 2lindisc(A_{k-1}) - a_k$.

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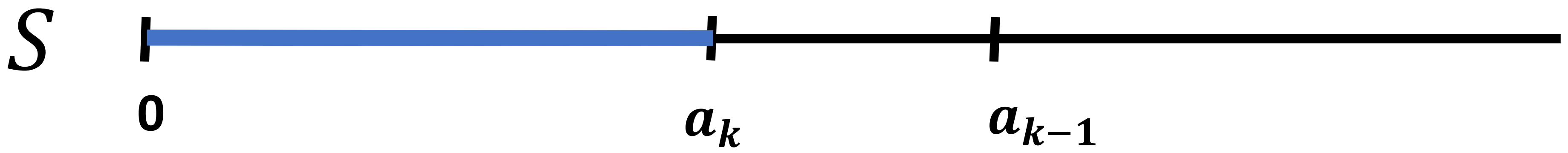

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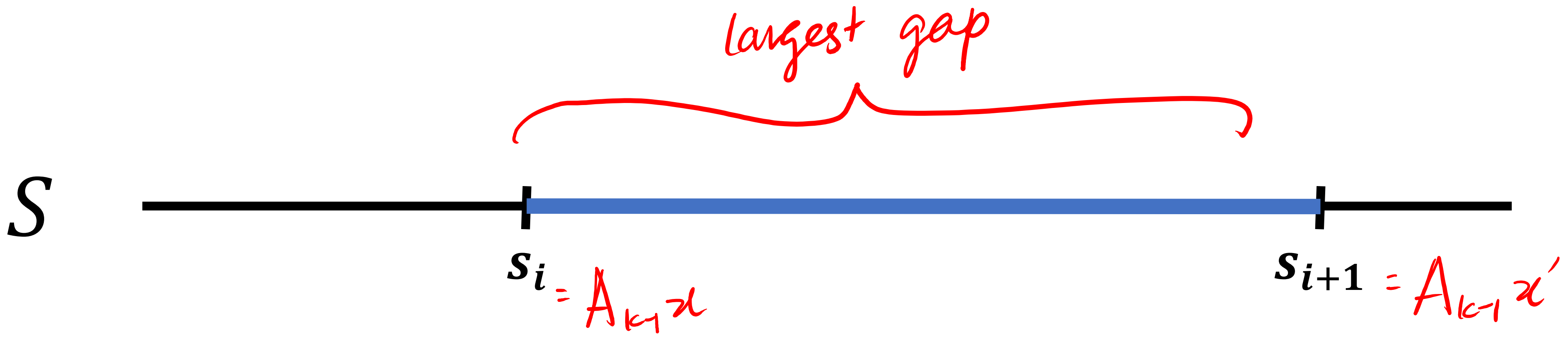

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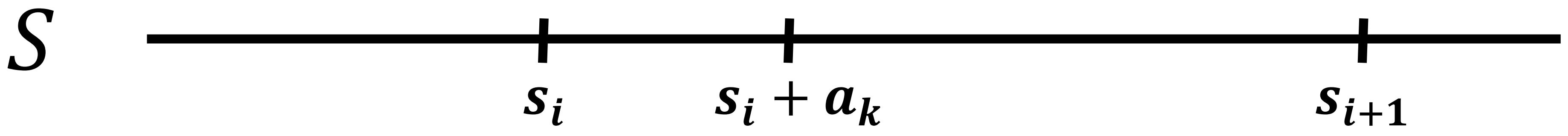

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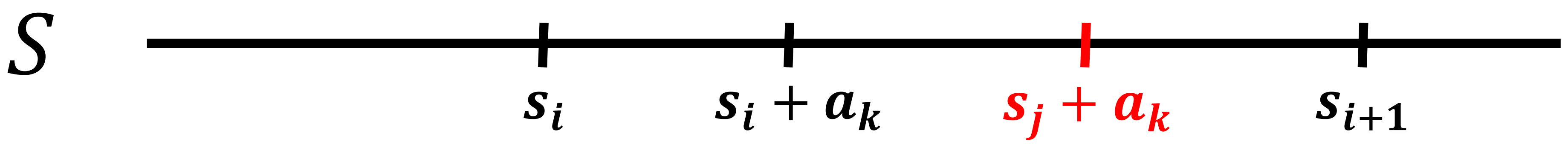

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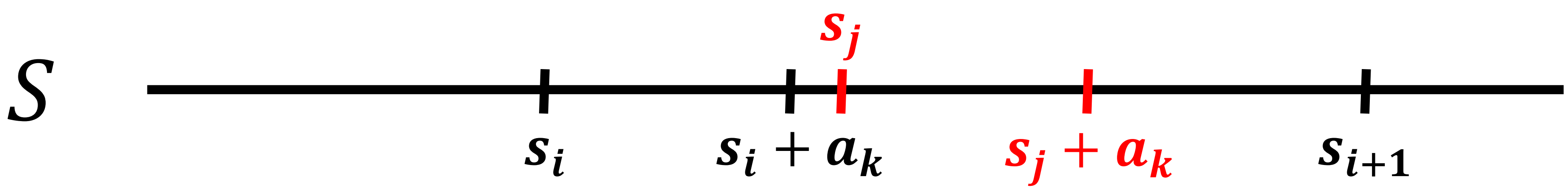

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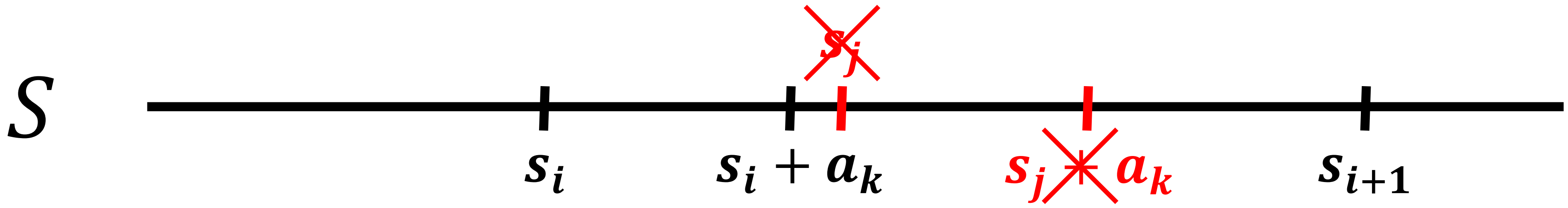

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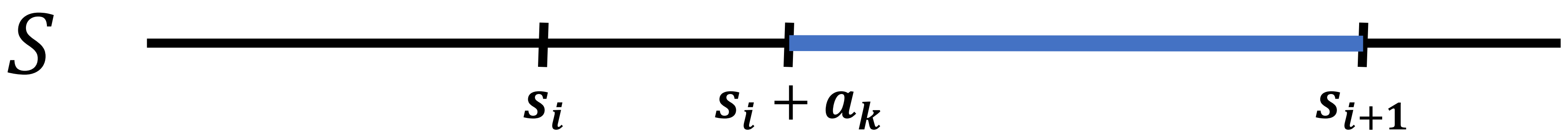

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CRP on Lattices:
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Other results:

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Other results:

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- $M \in \mathbb{Z}^{d\times n}$ with entries of magnitude bounded by δ; exact linear discrepancy in time

$$O\left(d(n\delta)^{d^2+d}\right)$$

- $M \in \mathbb{Z}^{m\times n}$; 2^{n+1} approximation in polynomial time

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